Journal of Sound and Vibration (2001) **248**(1), 187–194
doi:10.1006/jsvi.2001.3685, available online at http://www.idealibrary.com on IDEAL®

THE EFFECT OF THE POISSON RATIO ON THE VIBRATION OF HOLLOW CIRCULAR FINITE-LENGTH CYLINDERS

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(*Received* 1 *March* 2001, *and in final form* 28 *March* 2001)

1. INTRODUCTION

A recent report by Leissa and So [1] recorded results for the frequencies of vibration for solid circular cylinders with free–free boundary conditions. That report included the results concerning the effect of the Poisson ratio on the frequency of vibration for free–free cylinders. The present study is intended to extend those results to include hollow cylinders. An investigation of the frequency of vibration for hollow cylinders with free–free boundary conditions was reported by So and Leissa [2] and included several different hollow cylinder sizes with the Poisson ratio of $\nu = 0{\cdot}3$. The results given in reference [2] serve to validate the current analysis.

2. ANALYSIS

A finite element analysis [3] was developed to facilitate computing frequencies for solid cylinders. Frequencies and mode shapes were reported for a variety of solid cylinders with various boundary conditions. In this report, the analysis that was developed in reference [3] is used to analyze hollow cylinders with free–free and fixed–fixed boundary conditions. The finite element will not be discussed here since the development of the basic element is available [3]. It is useful to note that the element is based upon three-dimensional elasticity in cylindrical co-ordinates. The element–co-ordinate system is rendered axisymmetrically two-dimensional by assuming a solution that satisfies the governing equations in the circumferential θ direction [3]:

$$u(r, z, \theta, t) = U(r, z)\cos m\theta \cos \omega t, \qquad v(r, z, \theta, t) = V(r, z)\sin m\theta \cos \omega t,$$

$$w(r, z, \theta, t) = W(r, z)\cos m\theta \cos \omega t, \tag{1}$$

where m is the circumferential wave number and ω is the circular frequency. Solutions are computed for integer values of m. The final form of the eigenvalue problem is written as

$$[\mathbf{K}]\,\{\mathbf{u}\} - \omega^2\,[\mathbf{M}]\,\{\mathbf{u}\} = 0, \tag{2}$$

where $[\mathbf{K}]$ is the stiffness matrix, $[\mathbf{M}]$ is the mass matrix and $[\mathbf{u}]$ is the eigenvector.

0022-460X/01/460187 + 08 \$35.00/0

Table 1

Non-dimensional frequency Ω for free–free hollow cylinder with $L/a = 2$ and $b/a = 0{\cdot}2$

m	Mode	$\nu = 0{\cdot}0$	$\nu = 0{\cdot}1$	$\nu = 0{\cdot}2$	$\nu = 0{\cdot}3$	$\nu = 0{\cdot}4$
0†	1	1·981(6)	2·271(7)	2·283(6)	2·285(6)	2·287(6)
	2	2·223(8)	2·362(8)	2·530(8)	2·695(8)	2·853(8)
	3	2·258(9)	2·505(9)	2·683(9)	2·870(10)	3·066
	4	2·404(10)	2·680(10)	3·056	3·478	3·557
	5	3·162	3·285	3·390	3·537	4·086
0‡	1	1·571(1)	1·571(1)	1·571(1)	1·571(1)	1·571(1)
	2	3·142	3·142	3·142	3·142	3·142
	3	4·715	4·715	4·715	4·715	4·715
	4	5·223	5·223	5·223	5·223	5·223
	5	5·454	5·454	5·454	5·454	5·454
1	1	1·793(4)	1·871(4)	1·935(4)	1·986(4)	2·015(4)
	2	1·950(5)	1·972(5)	1·990(5)	2·003(5)	2·028(5)
	3	2·501	2·614	2·720(10)	2·813(9)	2·887(9)
	4	2·860	2·936	3·001	3·056	3·102
	5	2·947	3·026	3·117	3·222	3·333
2	1	1·665(2)	1·690(2)	1·714(2)	1·737(2)	1·760(2)
	2	1·775(3)	1·817(3)	1·860(3)	1·904(3)	1·950(3)
	3	2·187(7)	2·245(6)	2·298(7)	2·347(7)	2·392(7)
	4	2·822	2·895	2·956	3·009	3·056(10)
	5	3·724	3·817	3·876	3·928	3·957
3	1	3·129	3·139	3·148	3·155	3·162
	2	3·224	3·262	3·288	3·307	3·322
	3	3·421	3·462	3·505	3·547	3·588
	4	3·815	3·923	4·013	4·087	4·150
	5	4·613	4·736	4·775	4·805	4·829
4	1	4·233	4·246	4·253	4·258	4·262
	2	4·234	4·265	4·286	4·299	4·310
	3	4·557	4·617	4·674	4·726	4·773
	4	4·752	4·878	4·985	5·074	5·149
	5	5·397	5·550	5·677	5·731	5·751

† Pure radial/longitudinal.
‡ Pure torsional mode.

3. RESULTS FOR FREQUENCY

The hollow cylinder is assigned a length L, outside radius a and inside radius b. Results are given in terms of unit outside radius $a = 1$, unit shear modulus $G = 1$, unit mass density ρ and cylinder height $L = 2a$. The frequency is reported, similar to references [1, 3], using a non-dimensional frequency Ω defined as

$$\Omega = \omega a \sqrt{\rho/G}. \tag{3}$$

The results for free–free boundary conditions are tabulated in Tables 1–3 and results for fixed–fixed boundary conditions are given in Tables 4–6. Frequencies are tabulated for five different values of the Poisson ratio. Frequencies that correspond to the Poisson ratio of 0·3

TABLE 2

Non-dimensional frequency Ω for free–free hollow cylinder with $L/a = 2$ and $b/a = 0{\cdot}5$

m	Mode	$\nu = 0{\cdot}0$	$\nu = 0{\cdot}1$	$\nu = 0{\cdot}2$	$\nu = 0{\cdot}3$	$\nu = 0.3$ [2, 4]	$\nu = 0{\cdot}4$
0†	1	1·841(7)	1·994(7)	2·011(7)	2·042(7)	2·043	2·050(7)
	2	1·913(8)	2·014(8)	2·030(8)	2·148(8)	2·151	2·200(8)
	3	1·975(9)	2·036(9)	2·168(9)	2·302(10)	2·305	2·395(10)
	4	2·221	2·378	2·606	2·893	2·893	3·220
	5	2·793	2·847	2·971	3·093	3·091	3·242
0‡	1	1·571(4)	1·571(4)	1·571(3)	1·571(3)	1·571	1·571(3)
	2	3·142	3·142	3·142	3·142	3·142	3·142
	3	4·715	4·715	4·715	4·715	4·712	4·715
	4	6·293	6·293	6·293	6·283	6·283	6·293
	5	6·815	6·815	6·815	6·815	6·814	6·815
1	1	1·544(3)	1·568(3)	1·588(4)	1·604(4)	1·604	1·618(4)
	2	1·686(5)	1·762(5)	1·831(5)	1·894(5)	1·893	1·950(5)
	3	2·440	2·456	2·467	2·481	2·481	2·497
	4	2·462	2·648	2·859	2·931	2·931	2·992
	5	2·715	2·791	2·862	3·079	3·079	3·301
2	1	0·849(1)	0·887(1)	0·927(1)	0·971(1)	0·970	1·018(1)
	2	0·962(2)	0·990(2)	1·017(2)	1·046(2)	1·045	1·073(2)
	3	1·774(6)	1·829(6)	1·883(6)	1·936(6)	1·935	1·990(6)
	4	2·360	2·414	2·459	2·498	2·498	2·533
	5	3·142	3·206	3·254	3·294	3·293	3·330
3	1	2·056(10)	2·131(10)	2·208(10)	2·288(9)	2·287	2·370(9)
	2	2·180	2·238	2·294	2·349	2·348	2·402
	3	2·614	2·677	2·740	2·805	2·803	2·873
	4	3·229	3·318	3·40	3·478	3·477	3·551
	5	3·951	4·048	4·130	4·201	4·199	4·266
4	1	3·370	3·469	3·568	3·662	3·659	3·740
	2	3·473	3·549	3·618	3·682	3·680	3·747
	3	3·735	3·806	3·877	3·952	3·950	4·038
	4	4·170	4·278	4·382	4·483	4·482	4·584
	5	4·804	4·936	5·055	5·162	5·158	5·261

† Pure radial/longitudinal.
‡ Pure torsional mode.

are compared with the results obtained by Leissa and So [2] and So [4]. The inside radius of the cylinder is b and results are given for b/a ratios of 0·2, 0·5 and 0·9.

Finite element analysis can be sensitive to the aspect ratio of the element. Results for b/a of 0·2 and 0·5 were computed using a 50-element model with aspect ratios of 1·25 and 2·00 respectively. Tables 3 and 6 for b/a of 0·9 were developed using a 36- element model with an element aspect ratio of 5·00. A double-precision eigenvalue routine was used for the 36-element model.

The numbers in parentheses show the order of the frequencies and it can be seen that the wall thickness of the cylinder has a definite effect on the order of frequencies. The first frequency for free–free cylinders with inside radius $b = 0{\cdot}2$ is the first pure torsional mode

TABLE 3

Non-dimensional frequency Ω for free–free hollow cylinder with L/a = 2 and b/a = 0·9

m	Mode	$\nu = 0·0$	$\nu = 0·1$	$\nu = 0·2$	$\nu = 0·3$	$\nu = 0·3$ [2, 4]	$\nu = 0·4$
0†	1	1·488	1·557	1·608	1·647	1·647	1·675
	2	1·490	1·560	1·627	1·691	1·691	1·750
	3	1·499	1·570	1·641	1·707	1·707	1·771
	4	1·600	1·675	1·750	1·824	1·823	1·899
	5	1·887	1·978	2·073	2·173	2·168	2·282
0‡	1	1·571	1·571	1·571	1·571	1·571	1·571
	2	3·142	3·142	3·412	3·142	3·142	3·142
	3	4·714	4·714	4·714	4·714	4·712	4·714
	4	6·288	6·288	6·288	6·288	6·283	6·288
	5	7·869	7·869	7·869	7·869	7·854	7·869
1	1	1·223(12)	1·251(12)	1·274(12)	1·294(12)	1·294	1·312(11)
	2	1·305(13)	1·369(13)	1·429(13)	1·485(13)	1·485	1·538(13)
	3	1·555	1·628	1·699	1·770	1·769	1·843
	4	1·886	1·976	2·070	2·158	2·157	2·160
	5	2·088	2·148	2·156	2·170	2·165	2·280
2	1	0·122(1)	0·128(1)	0·134(1)	0·146(1)	0·143	0·152(1)
	2	0·171(2)	0·176(2)	0·180(2)	0·188(2)	0·185	0·192(2)
	3	0·964(8)	1·007(8)	1·049(8)	1·091(8)	1·090	1·131(8)
	4	1·398(15)	1·467	1·527	1·589	1·587	1·651(14)
	5	1·853	1·907	1·941	1·972	1·971	2·001
3	1	0·339(3)	0·357(3)	0·377(3)	0·401(3)	0·400	0.427(3)
	2	0·415(4)	0·431(4)	0·448(4)	0·467(4)	0·466	0·488(4)
	3	0·859(7)	0·893(7)	0·928(7)	0·966(7)	0·964	1·006(7)
	4	1·399	1·458(15)	1·519(15)	1·585(15)	1·583	1·659(15)
	5	1·978	2·063	2·155	2·258(6)	2·250	2·376
4	1	0·644(5)	0·677(5)	0·715(5)	0·758(5)	0·757	0.808(5)
	2	0·728(6)	0·760(6)	0·795(6)	0·834(6)	0·833	0·876(6)
	3	1·039(10)	1·080(9)	1·124(9)	1·174(9)	1·172	1·230(9)
	4	1·546	1·607	1·675	1·750	1·746	1·838
	5	2·157	2·244	2·342	2·454	2·445	2·584
5	1	1·028(9)	1·108(10)	1·139(10)	1·207(10)	1·207	1.285(10)
	2	1·114(11)	1·166(11)	1·223(11)	1·285(11)	1·284	1·354(12)
	3	1·386(14)	1·444(14)	1·507(14)	1·579(14)	1·577	1·662
	4	1·847	1·920	2·002	2·095	2·090	2·205
	5	2·449	2·546	2·655	2·782	2·772	2·931
6	1	1·485	1·560	1·644	1·740	—	1·850
	2	1·571	1·645	1·727	1·817	—	1·915
	3	1·829	1·909	1·996	2·095	—	2·210
	4	2·260	2·352	2·454	2·572	—	2·710
	5	2·841	2·953	3·080	3·226	—	3·401

† Pure radial/longitudinal.
‡ Pure torsional mode.

TABLE 4

Non-dimensional frequency Ω for fixed–fixed hollow cylinder with L/a = 2 and b/a = 0·2

m	Mode	$v = 0{\cdot}0$	$v = 0{\cdot}1$	$v = 0{\cdot}2$	$v = 0{\cdot}3$	$v = 0{\cdot}4$
0†	1	2·245(4)	2·335(4)	2·453(4)	2·576(4)	2·705(4)
	2	2·584(6)	2·794(6)	3·031(6)	3·342(10)	3·767(10)
	3	4·223	3·330	3·526	3·786	4·123
	4	4·444	4·342	4·450	4·591	4·862
	5	4·842	4·725	5·013	5·159	5·412
0‡	1	1·571(2)	1·571(2)	1·571(2)	1·571(2)	1·571(2)
	2	3·142(10)	3·142(10)	3·142(10)	3·142(7)	3·142(6)
	3	4·715	4·715	4·715	4·715	4·715
	4	5·454	5·454	5·454	5·454	5·454
	5	6·095	6·095	6·095	6·095	6·095
1	1	1·345(1)	1·372(1)	1·389(1)	1·408(1)	1·428(1)
	2	2·438(5)	2·511(5)	2·581(5)	2·647(5)	2·710(5)
	3	2·896(7)	2·999(7)	3·109(8)	3·165(8)	3·223(8)
	4	3·045(9)	3·079(9)	3·119(9)	3·226(9)	3·352(9)
	5	3·890	3·991	4·105	4·235	4·355
2	1	2·181(3)	2·230(3)	2·283(3)	2·339(3)	2·401(3)
	2	2·972(8)	3·029(8)	3·085(7)	3·140(6)	3·194(7)
	3	3·878	3·911	3·947	3·987	4·034
	4	3·958	4·062	4·168	4·278	4·390
	5	4·283	4·394	4·499	4·600	4·702
3	1	3·606	3·655	3·701	3·744	3·787
	2	4·043	4·092	4·144	4·192	4·235
	3	4·817	4·847	4·878	4·911	4·948
	4	4·907	4·500	5·085	5·162	5·234
	5	5·380	5·538	5·688	5·823	5·944
4	1	4·687	4·753	4·811	4·864	4·913
	2	5·025	5·097	5·159	5·213	5·260
	3	5·710	5·782	5·810	5·840	5·873
	4	5·755	5·802	5·896	5·947	6·042
	5	6·511	6·601	6·679	6·749	6·819

† Pure radial/longitudinal.
‡ Pure torsional mode.

with the second and third frequency corresponding to $m = 2$. As the wall thickness decreases, the torsional mode is less dominant and the fundamental frequency corresponds to the circumferential wave number $m = 2$. The results in Tables 3 and 6 are extended to include additional circumferential wave numbers because the lower frequencies for thinner cylinders include larger circumferential wave numbers. The fixed–fixed cylinders show a behavior similar to that of free–free cylinders with the fundamental frequency corresponding to $m = 1$ for wall thickness of $b = 0{\cdot}2$ and $0{\cdot}5$. Table 6 shows that for $b = 0{\cdot}9$ the first frequency corresponds to $m = 3$ except for $v = 0{\cdot}4$ and $m = 2$ for that case. Additional results for hollow cylinders have been given by Yii [5].

TABLE 5

Non-dimensional frequency Ω for fixed–fixed hollow cylinder with L/a = 2 and b/a = 0·5

m	Mode	$v = 0·0$	$v = 0·1$	$v = 0·2$	$v = 0·3$	$v = 0·4$
0†	1	2·084(4)	2·197(4)	2·325(4)	2·478(4)	2·627(4)
	2	2·225(5)	2·331(5)	2·434(5)	2·531(5)	2·672(6)
	3	2·702(10)	2·822(10)	2·979(10)	3·193	3·492
	4	3·742	3·489	3·952	4·057	4·175
	5	4·566	4·692	5·000	5·364	5·500
0‡	1	1·571(3)	1·571(3)	1·571(3)	1·571(2)	1·571(2)
	2	3·142	3·142	3·142	3·142(10)	3·142(10)
	3	4·715	4·715	4·715	4·715	4·715
	4	6·293	6·293	6·293	6·293	6·293
	5	6·993	6·993	6·993	6·993	6·993
1	1	1·240(1)	1·262(1)	1·285(1)	1·308(1)	1·334(1)
	2	2·315(6)	2·400(6)	2·483(6)	2·561(6)	2·636(5)
	3	2·673(9)	2·716(9)	2·765(9)	2·8429(9)	2·898(9)
	4	2·785	2·892	3·014	3·157	3·331
	5	3·677	3·778	3·877	3·973	4·071
2	1	1·435(2)	1·483(2)	1·535(2)	1·594(3)	1·665(3)
	2	2·485(8)	2·537(8)	2·602(8)	2·671(8)	2·744(8)
	3	3·452	3·485	3·521	3·562	3·611
	4	3·776	3·868	3·961	4·055	4·153
	5	3·968	4·102	4·242	4·390	4·550
3	1	2·381(7)	2·461(7)	2·550(7)	2·650(7)	2·762(7)
	2	3·167	3·252	3·342	3·439	3·542
	3	4·270	4·372	4·476	4·533	4·566
	4	4·463	4·484	4·507	4·584	4·694
	5	5·220	5·328	5·426	5·520	5·616
4	1	3·589	3·696	3·809	3·931	4·062
	2	4·180	4·289	4·403	4·521	4·645
	3	5·071	5·190	5·310	5·431	5·554
	4	5·533	5·550	5·567	5·587	5·612
	5	6·193	6·326	6·424	6·481	6·546

† Pure radial/longitudinal.
‡ Pure torsional mode.

4. CONCLUDING REMARK

Finite-length hollow circular cylinders have been analyzed for free–free and fixed–fixed symmetrical boundary conditions with the Poisson ratio as the primary variable. A primary conclusion is that the frequencies increase as the Poisson ratio increases for all the cylinders that were studied.

TABLE 6

Non-dimensional frequency Ω for fixed–fixed hollow cylinder with $L/a = 2$ and $b/a = 0.9$

m	mode	$\nu = 0.0$	$\nu = 0.1$	$\nu = 0.2$	$\nu = 0.3$	$\nu = 0.4$
0†	1	1·505(10)	1·583(11)	1·670(11)	1·769(11)	1·854(11)
	2	1·605(14)	1·678(13)	1·742(13)	1·799(12)	1·883(12)
	3	1·892	1·983	2·079	2·185	2·312
	4	2·221	2·343	2·484	2·642	2·820
	5	2·416	2·534	2·678	2·870	3·138
0‡	1	1·571(12)	1·571(10)	1·571(9)	1·571(9)	1·571(8)
	2	3·142	3·142	3·142	3·142	3·142
	3	4·715	4·715	4·715	4·715	4·715
	4	6·292	6·292	6·292	6·292	6·292
	5	7·869	7·869	7·869	7·869	7·869
1	1	0·969(4)	0·991(4)	1·013(4)	1·034(4)	1·056(4)
	2	1·449(9)	1·518(9)	1·584(10)	1·649(10)	1·716(10)
	3	1·845	1·931	2·020	2·115	2·225
	4	2·384	2·475	2·552	2·633	2·723
	5	2·426	2·507	2·604	2·723	2·874
2	1	0·678(2)	0·698(2)	0·718(2)	0·741(2)	0·766(1)
	2	1·227(7)	1·280(7)	1·335(7)	1·395(7)	1·463(7)
	3	1·775(16)	1·857(16)	1·944(16)	2·042(15)	2·159(15)
	4	2·415	2·526	2·649	2·790	2·964
	5	2·954	2·988	3·024	3·061	3·101
3	1	0·636(1)	0·663(1)	0·693(1)	0·729(1)	0·773(2)
	2	1·156(5)	1·206(5)	1·262(5)	1·326(5)	1·404(5)
	3	1·771(15)	1·852(15)	1·942(15)	2·045(16)	2·173(16)
	4	2·477	2·590	2·716	2·863	3·046
	5	3·313	3·459	3·624	3·784	3·806
4	1	0·824(3)	0·865(3)	0·912(3)	0·969(3)	1·040(3)
	2	1·273(8)	1·333(8)	1·401(8)	1·484(8)	1·581(9)
	3	1·887	1·974	2·073	2·190	2·335
	4	2·620	2·739	2·874	3·032	3·230
	5	3·476	3·628	3·801	4·004	4·258
5	1	1·164(6)	1·223(6)	1·292(6)	1·375(6)	1·476(6)
	2	1·550(11)	1·625(12)	1·712(12)	1·816(13)	1·945(13)
	3	2·131	2·231	2·345	2·481	2·650
	4	2·858	2·988	3·136	3·311	3·528
	5	3·715	3·876	4·061	4·279	4·458
6	1	1·602(13)	1·682(14)	1·776(14)	1·889(14)	2·026(14)
	2	1·946	2·040	2·150	2·281	2·442
	3	2·487	2·604	2·739	2·898	3·096
	4	3·190	3·334	3·500	3·695	3·937
	5	4·031	4·206	4·406	4·612	4·931

† Pure radial/longitudinal.
‡ Pure torsional mode.

ACKNOWLEDGMENTS

The authors acknowledge the financial support that was provided by the Electric Power Center, Tennessee Technological University for C. B-Y. Yii during her graduate study at Tennessee Tech.

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